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Cusp solitons, shock waves and envelope solitons in a new non-linear transmission line

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Abstract. A new non-linear evolution equation is derived in the continuum limit of a dispersive non-linear transmission line. Since this equation has a similar structure to the Boussinesq equation, but with the non-linear term of higher-order derivatives, it can be called the derivative Boussinesq equation. This equation bears a cusp soliton. A solution for the voltage signal exhibits a shock-wave front. The asymptotic behaviour of this equation is related to the non-linear Schrödinger equation by the reductive perturbation method. Its solitary wave solution is expressed in terms of the bright-envelope soliton. Hence, the non-linear transmission line proposed in the present paper describes the density depression, the collisionless shock wave in plasmas and the modulation instability of the asymptotic wave propagating in this line.

1. Introduction

Non-linear waves in different branches of physics have been investigated in detail through theory and experiments. In particular, since the study of the non-linear transmission line, which has primarily interested us, was pioneered by Hirota and Suzuki (1970), various kinds of non-linear wave propagation have been discussed (Yagi and Noguchi 1976, Yagi *et al* 1978, Nagashima 1979, Yoshinaga and Kakutani 1980, Brugarino and Pantano 1983, Watanabe 1984, Nejoh 1985). Amongst these studies, studies on non-linear waves in plasmas have been performed by Lonngren *et al* (1975), Kiyashko *et al* (1975) and Nejoh (1985). In the conventional study, the solitary wave pulses have been experimentally observed by considering the non-linear capacitor parallel to the shunt branch of the line. In such transmission lines, however, we cannot discuss the higher-order non-linear interaction in strong dispersive media.

The principal object of this paper is to show analytically the stationary solutions and the asymptotic solution of the new non-linear evolution equation, and to discuss the comparison between the waves propagating in this system and the ion acoustic waves in collisionless plasmas, by referring to a new non-linear transmission line as a lossless physical system with a new non-linear interaction in strong dispersive media. The non-linear transmission line introduced in this paper is different from that which Lonngren *et al* treated, in the following sense. That is, we introduce the non-linear capacitor parallel to the series branch and also consider the higher-order dispersion effect. The non-linear transmission line gives rise to a new non-linear evolution equation from its conservation laws. The non-linear equation derived is similar to the Boussinesq equation (1877), but the non-linear term is characterised by the higher-order derivatives

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compared with those of the Boussinesq equation. This equation is not only a mathematical model for a system in which the higher-order derivative non-linearity and the strong dispersion coexist, but is very important in physical systems. We analyse the stationary solution of this equation and determine the solution for the voltage signal propagating through this transmission line. The asymptotic solution and its instability in this new equation can be described by the reductive perturbation method (Taniuti 1974).

The layout of this paper is as follows. The conservation laws of the non-linear transmission line give rise to a new non-linear evolution equation in the continuum limit (§ 2). A new stationary solitary wave solution of this equation and the solution for the voltage signal are analysed in § 3. The derivation of the non-linear Schrödinger equation and its modulational instability are presented in § 4. The comparison between the transmission line and the ion acoustic wave in collisionless plasmas is shown in § 5. Results concerning this work are given in § 6. The last section is devoted to the concluding discussion.

2. A new non-linear evolution equation

Let us consider the dispersive non-linear transmission line shown in figure 1. At the nth section, we obtain the conservation laws of the current and the charge as follows:

$$(\partial/\partial t)C_2 V_n = I_{n-1} - I_n \tag{1a}$$

$$L(\partial/\partial t)I'_n = V_n - V_{n+1} \tag{1b}$$

$$(\partial/\partial t)Q_1(V_n - V_{n+1}) = I_n - I'_n \tag{1c}$$

$$Q_1(V_n - V_{n+1}) = (V_n - V_{n+1})C_1(V_n - V_{n+1}) \tag{1d}$$

where V_n denotes the voltage across the nth capacitor C_2, C_2 is the constant capacitance parallel to the shunt branch, $C_1(V_n - V_{n+1})$ is the capacitance parallel to the series branch, I_n is the total line current through the nth section and I'_n is the current through the coil whose inductance is L. A section size of the line is assumed to be h. We introduce the non-linearity into the capacitor $C_1(V_n - V_{n+1})$ parallel to the series branch and consider the dispersion in both capacitors $C_1(V_n - V_{n+1})$ and C_2 in the line, which is different from Lonngren's transmission line. We assume that the non-linear capacitance $C_1(V_n - V_{n+1})$ depends on the input voltage $V_n - V_{n+1}$ for the nth capacitor in

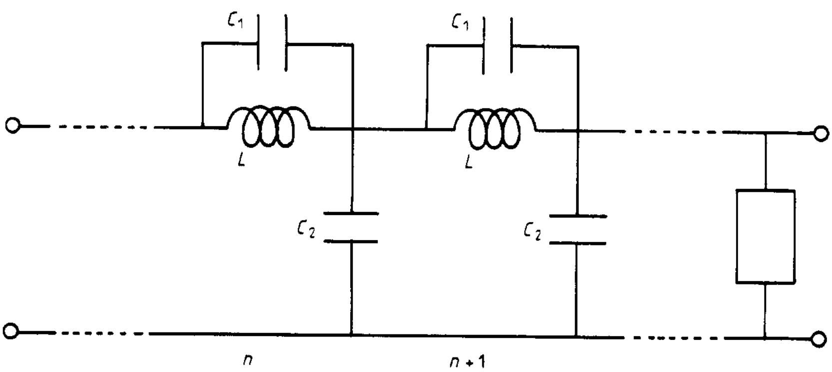

Figure 1. The nth and $(n+1)$th sections of a non-linear transmission line model.

the series branch in the line. The stored charge in the nth non-linear capacitor is expressed in terms of equation ($1d$). In addition, we assume the logarithmic non-linearity for the capacitance $C_1(V_n - V_{n+1})$ to be

$$C_1(V_n - V_{n+1}) = C_1 V_o \frac{\log[1 + (V_n - V_{n+1})/V_0]}{V_n - V_{n+1}} \tag{2}$$

where C_1 and V_0 take constant values. In this model the non-linear function of the bias and signal voltages across the series branch can be described by using a reverse biased *pn* junction diode for the capacitors. Eliminating the currents from equations ($1a$)-($1d$) and (2), we obtain a non-linear differential difference equation:

$$\frac{\partial^2 V_n}{\partial t^2} - \frac{1}{LC_2}(V_{n-1} - 2V_n + V_{n+1}) - \frac{C_1}{C_2} V_0 \frac{\partial^2}{\partial t^2} \log\left(\frac{1 + (V_{n-1} - V_n)/V_0}{1 + (V_n - V_{n+1})/V_0}\right) = 0. \tag{3}$$

The first two terms of the left-hand side of equation (3) represent the linear characteristics of the line. When we approximate the logarithmic non-linearity by its expansion up to the second term as follows:

$$\log[1 + (V_n - V_{n+1})/V_0] = (V_n - V_{n+1})/V_0 - \tfrac{1}{2}[(V_n - V_{n+1})/V_0]^2 \tag{4}$$

we can reduce equation (4) in the continuum limit to the following equation:

$$\frac{\partial^2 U}{\partial T^2} - \left(\frac{\partial^2 U}{\partial X^2} + \frac{1}{12}\frac{\partial^4 U}{\partial X^4}\right) - \frac{C_1}{C_2}\frac{\partial^4}{\partial T^2 \partial X^2}(U + \tfrac{1}{2}U^2) = 0. \tag{5}$$

Equation (5) is valid under the condition of $|U| < 1$. The variables T and X are defined by

$$T = t/(LC_2)^{1/2} \tag{6a}$$

$$X = x/h \tag{6b}$$

and the function U is defined by

$$U = \frac{1}{V_0}\frac{\partial V}{\partial X}. \tag{6c}$$

Combining equations (5) and ($6c$), we discuss the stationary solutions in the next section.

3. Stationary solutions

Introducing a moving coordinate ξ with velocity s as follows:

$$\xi = X - sT \tag{7}$$

we seek a stationary solution of equation (5) under the boundary conditions of

$$U \to 0 \qquad \partial^n U/\partial \xi^n \to 0$$

with $n = 1$ and 2 at $|\xi| \to \infty$. After applying the transformation of equation (7), we carry out the integration of equation (5) twice to obtain

$$(\mathrm{d}^2/\mathrm{d}\xi^2)(bU + cU^2) - aU = 0 \tag{8}$$

where the constants a, b and c are defined as

$$a = s^2 - 1 \tag{9a}$$

$$b = \tfrac{1}{12} + (C_1/C_2)s^2 \tag{9b}$$

$$c = \tfrac{1}{2}(C_1/C_2)s^2. \tag{9c}$$

We notice that the constants b and c are positive, while the sign of the constant a can be positive or negative, depending on whether the wave is supersonic or subsonic.

Multiplying $\mathrm{d}(bU + cU^2)/\mathrm{d}\xi$ on both sides of equation (8), we integrate to obtain

$$\tfrac{1}{2}[(\mathrm{d}/\mathrm{d}\xi)(bU + cU^2)]^2 = \tfrac{1}{2}abU^2 + \tfrac{2}{3}acU^3. \tag{10}$$

In order to ensure that the right-hand side of equation (10) is positive under the condition of $|U| < 1$, the constant a must be positive. Thus only the supersonic wave can propagate as a stationary signal. Equation (10) is reduced to the following differential expression:

$$\pm\mathrm{d}\xi = \frac{1}{2}\left(\frac{3}{ac}\right)^{1/2} \frac{b + 2cU}{[U^2(U + 3b/4c)]^{1/2}}\,\mathrm{d}U. \tag{11}$$

Here the factor $b + 2cU$ is a positive quantity under the condition of $|U| < 1$. An elementary integration of equation (11) yields

$$\begin{aligned}\pm\tfrac{1}{3}(a/b)^{1/2}(\xi - \xi_0) = {} & \tfrac{2}{3}\tanh^{-1}[1 + (4cU/3b)(\xi - \xi_0)]^{1/2} - [1 + (4cU/3b)(\xi - \xi_0)]^{1/2} \\ & - \tfrac{2}{3}\tanh^{-1}(1 + 4cU_0/3b)^{1/2} + (1 + 4cU_0/3b)^{1/2}\end{aligned} \tag{12}$$

in the region $-\infty < \xi < +\infty$, where U_0 is the height of the solitary wave at the centre $\xi \to \xi_0$. In figure 2, we illustrate the stationary solitary wave solution $U(\xi - \xi_0)$ given by equation (12) for several values of the parameter U_0.

Next, returning to the definition of U given in equation (6c), we determine the voltage V at a distance X. The integration of (6c) yields for the dimensionless voltage $\mathcal{V}$

$$\begin{aligned}\mathcal{V}(\xi - \xi_0) \equiv \frac{V}{V_0} &= \int_{-\infty}^{X} U\,\mathrm{d}X = \int_{-\infty}^{\xi} U\,\mathrm{d}\xi = \int_0^U U\frac{\mathrm{d}\xi}{\mathrm{d}U}\,\mathrm{d}U \\ &= \frac{1}{2}\left(\frac{3}{ac}\right)^{1/2}\int_0^U (\pm 1)\,U\frac{b + 2cU}{[U^2(U + 3b/4c)]^{1/2}}\,\mathrm{d}U\end{aligned} \tag{13}$$

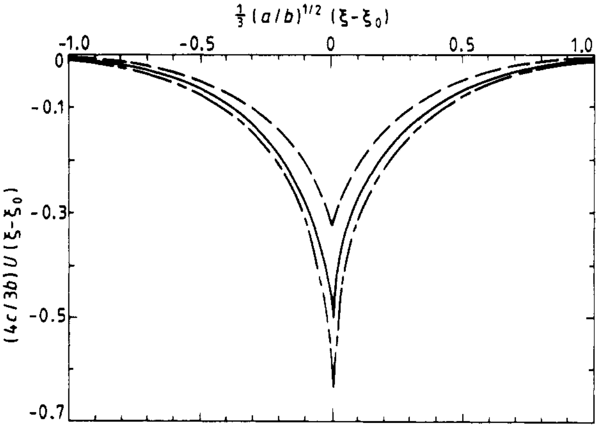

Figure 2. The cusp solitons $(4c/3b)U(\xi - \xi_0)$ against the coordinate $\frac{1}{3}(a/b)^{1/2}(\xi - \xi_0)$ for $(4c/3b)U_0 = -\frac{2}{3}$ (- - -); for $(4c/3b)U_0 = -\frac{1}{2}$ (——), and for $(4c/3b)U_0 = -\frac{1}{3}$ (— - —).

where (-1) denotes the region where $-\infty<\xi<0$ and $(+1)$ denotes the region where $0<\xi<+\infty$. Thus we obtain

$$\mathcal{V}(\xi-\xi_0)=-\frac{3b}{4c}\left(\frac{b}{a}\right)^{1/2}\left(\frac{4c}{3b}U(\xi-\xi_0)\right)\left(1+\frac{4c}{3b}U(\xi-\xi_0)\right)^{1/2} \tag{14a}$$

in the region of $-\infty<\xi<0$, and

$$\mathcal{V}(\xi-\xi_0)=\frac{3b}{4c}\left(\frac{b}{a}\right)^{1/2}\left(\frac{4c}{3b}U(\xi-\xi_0)\right)\left(1+\frac{4c}{3b}U(\xi-\xi_0)\right)^{1/2}$$
$$-\frac{3b}{2c}\left(\frac{a}{b}\right)^{1/2}\left(\frac{4c}{3b}U_0\right)\left(1+\frac{4c}{3b}U_0\right)^{1/2} \tag{14b}$$

in the region of $0<\xi<+\infty$. In figure 3, we illustrate the voltage $\mathcal{V}(\xi-\xi_0)$ as a function of $(\xi-\xi_0)$ determined from equations (14*a*) and (14*b*) combined with equation (12) for several values of U_0.

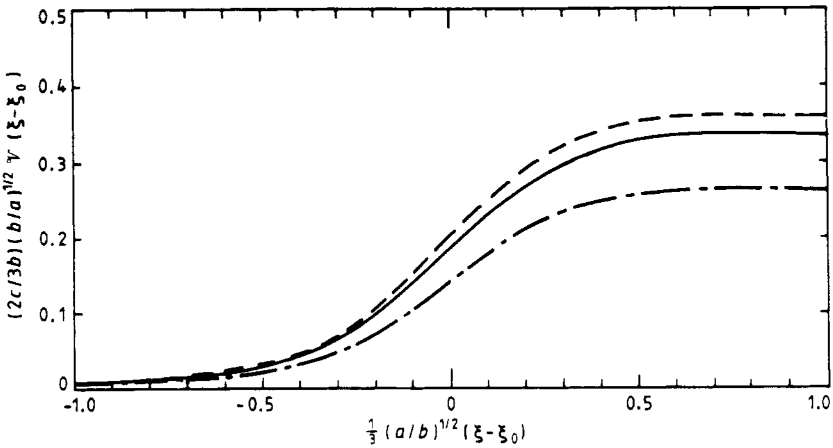

Figure 3. The curves of the normalised voltage signal $(2c/3b)(b/a)^{1/2}\mathcal{V}(\xi-\xi_0)$ against the coordinate $\frac{1}{3}(a/b)^{1/2}(\xi-\xi_0)$ for $(4c/3b)U_0=-\frac{2}{3}$ (- - -); for $(4c/3b)U_0=-\frac{1}{2}$ (——), and for $(4c/3b)U_0=-\frac{1}{3}$ (— - —).

4. Asymptotic behaviour

We analyse the asymptotic behaviour of the slowly varying finite-amplitude wave in the strong dispersive region of equation (5) by using the reductive perturbation method. We assume, according to the fundamental philosophy of this method, that the temporal asymptotic behaviour of the wave is slower than the spatial distortion of the wave and that the wave propagates at the group velocity. We introduce the stretched coordinates τ and η, defined as

$$\tau=\varepsilon^2 T \tag{15a}$$

$$\eta=\varepsilon(X-\lambda T) \tag{15b}$$

where ε is a small parameter and λ is the group velocity $\partial\omega/\partial k$. The complex function U normalised by V_0 can be expanded in power series of ε as follows:

$$U=\sum_{n=0}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}U^{(n)}(l,\eta,\tau)\exp(\mathrm{i}l\theta) \tag{16}$$

where $\theta=kx-\omega t$ and l refers to the higher-harmonic wave components. The complex function $U^{(n)}(l)$ satisfies the reality condition

$$U^{(n)}(l)=U^{(n)}(-l)^*. \tag{17}$$

The asterisk denotes complex conjugation. Substituting equations (15a), (15b) and (16) into the new non-linear evolution equation (5) and equating the coefficients of each order of ε to zero, we get the zeroth-, first- and second-order sets of equations. We have the following linear dispersion relation for $l=\pm1$ of the zero-order equation:

$$\omega^2=\frac{k^2(1-\frac{1}{12}k^2)}{1+(C_1/C_2)k^2} \tag{18}$$

where ω is normalised by $(LC_2)^{-1/2}$, because $U^{(0)}(\pm1)$ is non-trivial. The $l=\pm1$ components of the first-order equation give rise to the group velocity

$$\lambda=\frac{\partial\omega}{\partial k}=\frac{k(1-\frac{1}{6}k^2)-(C_1/C_2)k\omega^2}{\omega[1+(C_1/C_2)k^2]}. \tag{19}$$

We obtain the non-linear Schrödinger equation for the $l=1$ component of the second-order perturbation equation of ε to be

$$\mathrm{i}\frac{\partial\psi}{\partial\tau}+P\frac{\partial^2\psi}{\partial\eta^2}+Q|\psi|^2\psi=0 \tag{20}$$

with the definition of

$$P=-\frac{1}{2}\frac{k^2}{\omega^3[1+(C_1/C_2)k^2]} \tag{21a}$$

$$Q=-\frac{1}{3}\left(\frac{C_1}{C_2}\right)^2\frac{\omega^3}{1+\frac{1}{12}C_2/C_1} \tag{21b}$$

and

$$\psi=U^{(0)}(1). \tag{21c}$$

Here the coefficients (21a) and (21b) are abbreviated by the dispersion relation (18) and we have used the reality condition. Thus we notice that $P<0$ and $Q<0$ because L, C_1, C_2, k and ω are all positive. We express the complex amplitude $\psi(\eta,\tau)$ of equation (20) in terms of

$$\psi(\eta,\tau)=\rho^{1/2}\exp\left(\mathrm{i}\int^{\eta}\frac{\sigma}{2P}\,\mathrm{d}\eta\right)$$

and expand the real functions ρ and σ as follows:

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix}=\begin{pmatrix}\rho_0\\ \sigma_0\end{pmatrix}+\begin{pmatrix}\delta\rho\\ \delta\sigma\end{pmatrix}\exp[\mathrm{i}(K\eta-\Omega\tau)].$$

When $PQ>0$, the frequency shift Ω for the perturbations $\delta\rho$ and $\delta\sigma$ takes the form

$$\Omega=[\sigma_0\pm\mathrm{i}(2PQ\rho_0)]K.$$

Therefore, a plane-wave solution of equation (20) is modulationally unstable against the non-linear self-modulation of its amplitude and phase. By a brief calculation, we obtain the bright-envelope soliton solution

$$\psi(\eta, \tau) = A\,\mathrm{sech}[(QA^2/2P)^{1/2}\eta]\exp[\mathrm{i}(\tfrac{1}{2}QA^2)\tau]$$

where A is the amplitude. However, we have no solution as obtained in the Korteweg-de Vries soliton in the long wavelength limit.

5. Comparison between the line and the ion acoustic wave

In this section we discuss the correspondence between the variables of the transmission line and the ion acoustic wave. We obtain the linear equation for the voltage from equation (5) in the continuum limit. For small amplitude signals of the form $V \sim V_0\exp[\mathrm{i}(kx - \omega t)]$, we can derive the dispersion relation (18).

On the other hand, for one-dimensional ion acoustic waves travelling in a collisionless plasma composed of cold ions ($T_i = 0$) and isothermal electrons, we neglect the effects of the Landau damping and the electron inertia. When we describe the behaviour of the plasma by the two-fluid model without dissipation, it is well known that the dispersion relation of the wave propagating in this system is written in terms of

$$\omega^2 = k^2/(1+k^2). \tag{22}$$

Here, ω and k are normalised by $\omega_i = (4\pi n_0 e^2/M)^{1/2}$ and $k_D = (4\pi n_0 e^2/\kappa T_e)^{1/2}$, respectively, where n_0, M, κ, e and T_e denote the characteristic density, the ion mass, the Boltzmann constant, the electron charge and the electron temperature (T_e is assumed to be constant), respectively.

From equations (18) and (22) we compare the parameter of the line with that of ion acoustic waves. The correspondence between the line and the plasma can be compiled as follows:

Line	Plasma
V	u
L	M
I_n	n_e
I'_n	n_i
h^2/C_2	κT_e
$1/C_1$	$4\pi n_0 e^2$
$1/LC_1 = \omega_0^2$	$4\pi n_0 e^2/M = \omega_i^2$
$h^2/LC_2 = v_0^2$	$\kappa T_e/M = v_T^2$
$(C_1/C_2)h^2 = \delta^2$	$\kappa T_e/4\pi n_0 e^2 = \lambda_D^2 = k_D^{-2}$

Here $\omega_0 = (LC_1)^{-1/2}$, $v_0 = (LC_2)^{-1/2}h$ and $\delta = (C_1/C_2)^{1/2}h$, and they are the cut-off frequency, the low-frequency velocity and the characteristic length, respectively. u, n_e, n_i, ω_i, v_T and λ_D refer to the ion flow velocity, the electron density, the ion density, the ion plasma frequency, the characteristic ion acoustic velocity and the Debye length of the plasma, respectively. $\omega_i = (4\pi n_0 e^2/M)^{1/2}$, $v_T = (\kappa T_e/M)^{1/2}$ and $\lambda_D = (\kappa T_e/4\pi n_0 e^2)^{1/2}$. Therefore, the parameters of the line correspond to those of the plasma. However, the linear dispersion relation (18) of the line is markedly different from that of the ion acoustic wave (22). Furthermore, we note that the reciprocal of the capacitance parallel to the series branch of the line corresponds to the ion plasma density.

6. Results

A new non-linear transmission line proposed in this paper gives rise to a new non-linear evolution equation (5) in the continuum limit when we assume the logarithmic non-linearity for the capacitance in the series branch and consider the dispersion in both capacitors $C_1(V_n - V_{n+1})$ and C_2. We note that the non-linear term of this equation is a new one which is characterised by higher-order derivatives compared with that of the Boussinesq equation. We obtain analytically a cusp solitary wave as its stationary solution for the supersonic wave of this equation as illustrated in figure 2. In addition, as shown in figure 3, a stationary solution for the normalised voltage signal exhibits a shock-wave propagation. Furthermore, the non-linear Schrödinger equation describes the asymptotic behaviour of a new non-linear evolution equation (5) in the strong dispersive region by the reductive perturbation method. When the product of the dispersion coefficient and the non-linear coupling coefficient of the non-linear Schrödinger equation is positive for any wavenumber, its plane-wave solution is always modulationally unstable against the non-linear self-modulation. Hence, in this case, the plane-wave solution is described by the bright-envelope soliton.

The correspondence between the wave propagating in this line and the ion acoustic wave in collisionless plasmas has been discussed in the preceding section. The dispersion relation of this line (18) does not have one-to-one correspondence to that of the ion acoustic wave in plasmas (22). This is because for the former a term $\frac{1}{12}k^4$ in its numerator gives rise to a stronger dispersion at short wavelengths than for the ion acoustic wave. However, one-to-one correspondence holds between parameters of these two systems. In particular, we must note that the correspondence shown in this paper differs greatly from the correspondence that Lonngren *et al* (1975) showed between their line and the plasma, in the sense that the reciprocal of the capacitance in the series branch of this line corresponds to the ion plasma density.

7. Concluding discussion

We discuss the results obtained in this investigation as follows. We have found a new non-linear evolution equation in the continuum limit referring to a new non-linear transmission line proposed in this paper. Since the non-linear term of this equation is characterised by a higher-order derivative compared with that of the Boussinesq equation, we call this the derivative Boussinesq equation. First we showed that the derivative Boussinesq equation bears a cusp soliton as its stationary solution. Since the dependence of the non-linear capacitor on the input voltage corresponds to the non-linear self-interaction of the ion plasma density on the basis of the correspondence between the parameters of the line and the electrostatic ion acoustic wave in plasmas, we have theoretically found that the cusp soliton in this paper explains the density depression of the ion acoustic wave. Moreover, we have determined the voltage signal propagating through this line. It takes the form of the collisionless ion acoustic shock-wave front. Hence we propose that these two results describe the non-linear electrostatic ion acoustic shock-wave propagation in collisionless plasmas observed by Taylor *et al* (1970). Furthermore, it is also found that the asymptotic solution of the derivative Boussinesq equation can be expressed in terms of the bright-envelope soliton with modulational instability. Since the derivative Boussinesq equation is derived from the balance between a strongly dispersive term composed of four times differentiation and a new higher-order derivative non-linear term, the modulational instability of the asymptotic wave propagating in this line holds for all wavenumbers. Therefore, we cannot get such a solution as shown in the Korteweg–de Vries equation even in the limit of long wavelength.

We therefore conclude as follows. A cusp soliton, a shock-like wave front and the bright-envelope soliton of the proposed non-linear transmission line describe the density depression, the shock wave of the electrostatic wave in collisionless plasmas and the asymptotic behaviour of the derivative Boussinesq equation in the strong dispersive region, respectively. In particular, we emphasise that investigations of the cusp solitons which have interested us as mathematical models (Wadati *et al* 1980) will provide us with useful tools in understanding the properties of the physical system with higher-order derivative non-linear interaction in strong dispersive media. Since our analysis has been carried out for a new non-linear evolution equation, derived in the continuum limit, it would be worthwhile examining the analytical prediction described here in comparison with the experimental observation for the non-linear transmission line. The further application of the derivative Boussinesq equation to other physical systems is under investigation.

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